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A DIFFERENTIAL EQUATION FOR UNDAMPED FORCED NON-LINEAR OSCILLATIONS. III*

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by

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1. Introduction. The most general differential equation to which the dynamical description of the title applies is

$$\ddot{x} + g(x) = p(t), \tag{1.1}$$

where dots denote differentiation with respect to t. The essential problem for this equation is to determine the behaviour of solutions as $t \to \infty$. When we attack this problem, the most obvious question is whether every solution is bounded as $t \to \infty$; this question is open except when $g(x)$ is linear. (Moser's methods in (5) and (6) raise hopes that when $g(x)/x \to \infty$ it can be answered affirmatively.) In the special case when $p(t)$ is periodic, (1.1) may have periodic solutions; it is clear that any such solution is bounded, and it is worth mentioning that finding periodic solutions is the easiest way of finding particular bounded ones. So long as the boundedness problem is unsolved, there is a special interest in finding a large class of particular bounded solutions: if we know such a class then, although we cannot say whether the general solution is bounded or not, we can make the imprecise comment that

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either the general solution is in fact bounded *or* the structure of the whole set of solutions is quite complicated.

In two previous papers of the same title, (2) and (3), I have considered a tractable special case of (1.1). A heuristic argument given in (2) suggests that if $g(x)/x \to \infty$ as $|x| \to \infty$, (1.1) has all its solutions bounded and has some periodic solutions when $p(t)$ is periodic; by specializing $g(x)$ to be $2x^3$, we satisfy this condition and simplify our calculations as far as possible. If $p(t)$ is even, (1.1) is unaffected by changing t into $-t$; if $p(t)$ is both even and periodic, we obtain the simple criterion for a periodic solution given as Theorem A in §2. When we adopt both these specializations we are led to the special equation

$$\ddot{x} + 2x^3 = e(t); \tag{1.2}$$

here $e(t)$ is written to emphasize the assumed evenness, and we shall assume $e(t)$ has least period 2π. In this paper I continue the discussion of (1.2) and complete the work foreshadowed in (2).

In point of method, this paper's principal contribution is to develop a technique for making certain estimations: it will appear that we need to approximate certain partial derivatives connected with (1.2). We state a convenient form of these estimations as Theorem 7; it is not practicable to enunciate Theorem 7 until we reach §3, since we shall need to introduce notation.

u.c. script cee In (2) it was shown that (1.2) has an infinite class $\mathcal{C}$ of "large" periodic solutions whose periods are not "too large" compared with their amplitudes. Although these solutions could be described by giving the number of their extrema in a period they could not be uniquely characterized in this way. In this paper, under stronger hypotheses, we shall show that a large sub-class of $\mathcal{C}$ can be characterized in this way. Our calculations will lead us to a long statement (Theorem 8, enunciated in §3); it is more suitable here to give a simpler but typical result which we need in order to

ital l.c. omega enunciate Theorem 10. If, as before, we write ω for the constant defined by (2.3) and (2.4) below we can state:

THEOREM 9. For each large k there is, under the assumption that $e(t)$ has a continuous third derivative, just one solution $x_k(t)$ of (1.2) for which

(i) $\dot{x}_k(0) = 0$ and $x_k(t)$ has period 2π,

(ii) $x_k(t)$ has, in $0 \leq t < 2\pi$, k positive maxima, k negative minima and no other stationary points, and

(iii) $x_k(0) = k\omega + O(k^{-2})$ as $k \to \infty$.

Further, every solution of (1.2) which has period 2π, $\dot{x}(0) = 0$ and $x(0)$ large is an $x_k(t)$ for a suitable k.

In (3), by a process of interpolation between members of $\mathcal{C}$, it was shown that (1.2) has periodic solutions not in $\mathcal{C}$ whose periods may be arbitrarily long compared with their amplitudes. About individual solutions we obtained less information than about those of $\mathcal{C}$ but we did obtain results about the interlacing of solutions, most naturally set out in terms of fixed points of the topological transformation

$$T(a, b) = (x(2\pi;\ a,b;\ 0),\ \ \dot{x}(2\pi;\ a,b;\ 0)),$$

where $x(t;\ a,b;\ 0)$ is the solution of (1.2) with $x(0) = a$, $\dot{x}(0) = b$. If, as in (3), we write F_k for the point $(a_k, 0)$ of the (a, b)-plane, where $a_k = x_k(0)$, we can translate Theorem 9 into the language of transformations as:

COROLLARY to Theorem 9. For each large k there is, under the assumption that $e(t)$ has a continuous third derivative, just one fixed point F_k of T on the a-axis near $(k\omega, 0)$. Further, every fixed point of large abscissa on this axis is an F_k.

We can now state this paper's principal result on periodic solutions of (1.2):

THEOREM 10. Under the assumption that $e(t)$ has a continuous third derivative there is a constant positive integer C_5 such that, when k is large:

(i) for each m with $2 \leq m \leq (k - C_5)^2$, exactly $\varphi(m)$ fixed points of order m of T lie on the segment $F_k F_{k+1}$, say (from left to right)

$P_{1m}, \ldots, P_{sm}, \ldots, P_{m-1,m}$, where s runs through the numbers less than m and prime to it;

(ii) if s/m and s'/m' are irreducible fractions for which

$$0 \leq \frac{s}{m} < \frac{s'}{m'} \leq 1;$$

then P_{sm} lies to the left of $P_{s'm'}$, the convention $P_{01} = F_k$, $P_{11} = F_{k+1}$ being adopted; and

(iii) the solution (necessarily of least period $2m\pi$) to which P_{sm} corresponds has $mk + s$ positive maxima and $mk + s$ negative minima in this period.

It is important to point out that, in the quite long approximate calculations which lead to Theorem 7, we do not require the evenness of $e(t)$, and from this remark it follows that we may, without lengthening the discussion, choose an enunciation of Theorem 7 which applies to the equation

$$\ddot{x} + 2x^3 = p(t), \tag{1.3}$$

where $p(t)$ has least period 2π but is not necessarily even. Such an enunciation is adopted in §3. It is evident that the greater generality cannot be exploited in the papers of this series, since our criterion for periodic solutions of (1.2) depends directly on the evenness of $e(t)$. In a further paper

(4), however, I shall obtain a criterion for periodic solutions of (1.3) and then show that the estimates contained in Theorem 7 can be applied to prove that (1.3) also has an infinity of periodic solutions.

Except in §9, this paper does not refer to (3). Unlike (3), it requires that some of the detailed results of (2) should be quoted. A summary of these results and of the notations which they justify is given in §2; if these statements are accepted, this paper can be read independently of (2). Once the necessary notation from (2) has been re-introduced we can, in §3, enunciate Theorems 7 and 8. Also in §3 we describe the general plan of the calculations which occupy §§4 to 7 and lead to Theorem 7. In §8 we prove Theorems 8 and 9.

In §9 we establish Theorem 10 which depends on Theorems 5 and 6 of (3). At the beginning of §9 we give a summary of the results needed from (3); if these are accepted this section can be read independently of (3).

It is natural to ask how far the methods used in discussing (1.2) can be carried over to the equation

$$\ddot{x} + g(x) = e(t) \quad . \tag{1.4}$$

It is straightforward but tedious to check that the approximate calculations of (2) and this paper can be closely copied when $g(x)$ is a constant multiple of

$|x|^\alpha$ sgn x, α being greater than 2; the estimates so obtained can be applied to give results about periodic solutions, more or less detailed according as α is large or small. The more interesting questions are whether we can arrange our calculations satisfactorily under the conditions (occurring separately or together): (i) $g(x)$ does not always increase, (ii) $|g(x)|$ has very different rates of increase as $x \to \infty$ and $x \to -\infty$, and (iii) $g(x)/x$ increases very slowly. Harvey (1) has considered the case when $g(x)$ behaves as a polynomial for large $|x|$. Assuming that $g(x)$ is

$$\sum_{m=0}^{n_1} a_m x^m \quad \text{or} \quad \sum_{m=0}^{n_2} b_m |x|^m$$

for $x > R > 0$ or $x < -R$ respectively, and requiring that $n_1 > 1$, $n_2 > 1$, he shows that strong enough estimates can be obtained by the methods of (2) to deduce the existence of periodic solutions. Even in this polynomial case condition (ii) above shows itself as awkward to handle: to allow the possibility that $n_1 \neq n_2$ complicates Harvey's work considerably and when, in fact, $n_1 \neq n_2$, his estimates become weaker.

It is easier to state how far the methods of (3) could be applied to (1.4). If some periodic solutions had already been found by the methods used in (2), then, provided that $g(x)$ was strictly increasing, we should need only verbal changes in (3) to show that we could interpolate an infinity of periodic solutions between them, and to show that any interpolated solution was characterized by the number of maxima taken in a period by the difference between it and a standard periodic solution. If, however, $g(x)$ was not strictly increasing, this characterization would fail since Lemma 8 of (3) would no longer hold; it would still be possible to interpolate periodic solutions but the work in (3) would have to be modified in a number of points.

It will be convenient to continue the numbering of theorems from (2) and (3) but to begin a new numbering of lemmas, sections and equations.

2. Known properties of (1.2) and (1.3). We shall carry over the notational conventions of (2) so that, in particular, dashes are not used as symbols of differentiation and, to avoid having two suffixes with quite different meanings, special values of h_0 are labelled as h_0^* or $h_0^\dagger$, related solutions of (1.2) being labelled $x^*(t)$, $x^\dagger(t)$.

Although the estimations carried out in (2) were presented for (1.2), the evenness of $e(t)$ was not needed in the work. Without individual mention, we shall quote estimates from (2) so as to apply to (1.3), that is, we shall write $p(t)$ for $e(t)$ in them, and, correspondingly, we shall now write $x(t;\ a,b;\ t_0)$ for the solution of (1.3) with $x(t_0) = a$, $\dot{x}(t_0) = b$. This implies that we are assuming about $p(t)$:

(C1) $p(t)$ is not identically zero and has least period 2π,

(C2) $\int_0^{2\pi} p(t)dt = 0$, and

(C3) $p(t)$ is differentiable, $\dot{p}(t)$ is continuous, $|p(t)| < E$, $|\dot{p}(t)| < E$.

We can conveniently recall the notation and motivation of our earlier work by stating some simple properties of

$$\ddot{y} + 2y^3 = 0, \tag{2.1}$$

a differential equation which has the first integral

$$\dot{y}^2 + y^4 = h^4. \tag{2.2}$$

Evidently every solution of (2.2) is periodic with least period $2\varpi/h$, where we have written

$$\tfrac{1}{2}\varpi = \int_0^1 (1 - u^4)^{-\frac{1}{2}}\, du, \tag{2.3}$$

$-\frac{1}{4}$

and this is to say that the solution of (2.1) with $y(0) = a$ and $\dot{y}(0) = b$ is periodic with least period $2\varpi(b^2 + a^4)^{\frac{1}{4}}$. If m is a positive integer, there are solutions of (2.1) having $2m\pi$ as a period (not in general the least period); if we suppose k and s to be integers and write

curly pi

$$\omega = \varpi/\pi \tag{2.4}$$

we see, in particular, that the solution of (2.1) with $\dot{y}(0) = 0$ and $y(0) = (k + s/m)\omega$ has $2m\pi$ as a period. We shall be concerned with similarly determined solutions of (1.2) and (1.3).

Any solution of (1.3) has an infinity of stationary points. So long as a solution remains "large" its stationary values are successively positive maxima and negative minima; in such a succession we write $\varphi_0, \varphi_4, \varphi_8, \dots$ for the values of t giving the maxima and $\varphi_2, \varphi_6, \varphi_{10}, \dots$ for the values giving minima. We describe an interval $\varphi_{4\nu} \leq t \leq \varphi_{4\nu+4}$ as a cycle, and write $\varphi_{4\nu+1}, \varphi_{4\nu+3}$ for the intermediate values of t at which $x(t) = 0$. If $\varphi_0, \varphi_2, \dots, \varphi_{4n}$ are all defined we say that the solution describes n cycles in the interval $\varphi_0 \leq t \leq \varphi_{4n}$, and we extend this language by talking of half-cycles and then saying that the solution describes $\frac{1}{2}n$ cycles in $\varphi_0 \leq t \leq \varphi_{2n}$ or in $\varphi_{2n} \leq t \leq \varphi_{4n}$. Corresponding to any solution $x(t)$ of (1.3) we define $h^4(t) = \dot{x}^2(t) + x^4(t)$; we then have as an analogue of (2.2) the identity

That is, $\int_{t_0}^{t}$

$$h^4(t) = h^4(t_0) + 2\int_{t_0}^{t} p(t')\dot{x}(t')dt'. \tag{2.5}$$

[illegible]

Whenever φ_α is defined we define h_α as the positive number for which $h_\alpha^4 = h^4(\varphi_\alpha)$.

Consider a solution with a stationary value where $t = \varphi_0$, and write $x(\varphi_0) = h_0$. To express the requirement that the solution should be large enough for φ_2 and φ_4 to be defined, we introduce an r and prescribe that $h_0 > r$. To express the requirement that the solution should remain large for a considerable interval of t we have some choice of prescriptions. We

ll.c. rho

choose the form that, given a positive ρ, there is an $R = R(\rho)$ such that, when $h_o > R$, the functions $h_{2n} = h_{2n}(h_o, \varphi_o)$ and $\varphi_{2n} = \varphi_{2n}(h_o, \varphi_o)$ are defined for all φ_o and all $n \leq \rho h_o^3$, and, further, they are differentiable functions of h_o and φ_o. It is clearly open to us to discuss a large solution of (1.3) over a range of t by finding estimates for h_{2n} and φ_{2n} in terms of h_o and φ_o. Inside a cycle we have that, for large h_o,

$$\varphi_1 = \varphi_o + \tfrac{1}{2}\varpi h_o^{-1} + \underline{O}(h_o^{-4}) \tag{2.6}$$

and

$$h_1^4 = h_o^4 - 2p(\varphi_o)h_o + \underline{O}(1). \tag{2.7}$$

In a succession of cycles we have that, if $n \leq \rho h_o^3$, then

$$\varphi_{2n} = \varphi_o + \varpi n h_o^{-1} + \underline{O}(n^2 h_o^{-5}) + \underline{O}(n h_o^{-4}) \tag{2.8}$$

and, uniformly in $\varphi_o \leq t \leq \varphi_{2n}$,

$$h^4(t) = h_o^4 + \underline{O}(n) + \underline{O}(h_o), \tag{2.9}$$

where the constants implied in the $\underline{O}$-terms depend only on ρ, not on n.

We were concerned‡ in (2) and shall be concerned here with values of

‡Although (2.10) presents itself, in (2) and the present paper, only because we take an even $e(t)$, we introduce the estimation of h_o^* before discussing the evenness. This is desirable since the estimation applies with only trivial changes to the equation $\varphi_{4n} = 2m\pi$ which we shall meet in (4).

h_o and combinations of positive integers n and m for which

$$\varphi_{2n}(h_o, 0) = m\pi . \tag{2.10}$$

Evidently, when $n = mk + s$, (2.8) gives $m\pi$ as the principal term in $\varphi_{2n}((k + s/m)\omega, 0)$; if k is "large", m not "too large" compared with it and $s < m$ it is straightforward to use the estimations above to show that there is an h_o^* near $(k + s/m)\omega$ which satisfies (2.10), but it is essential to make the order of choice clear. A convenient order is to suppose a positive A assigned, determine a $C_4 = C_4(A)$ and a $k_5 = k_5(A)$, and then require $k > k_5$ and $0 \leq s < m \leq Ak^2$. With these hypotheses we show that $\mathcal{D}$ and $\mathcal{E}$, defined by

? script D, E

$$\mathcal{D} = \mathcal{D}(k, m, s; C_4) = (k + sm^{-1} - C_4\, mk^{-2})\omega$$

and

$$\mathcal{E} = \mathcal{E}(k, m, s; C_4) = (k + sm^{-1} + C_4\, mk^{-2})\omega$$

are such that

$$\varphi_{2n}(\mathcal{D}, 0) > m\pi \tag{2.11}$$

and

$$\varphi_{2n}(\mathcal{E}, 0) < m\pi; \tag{2.12}$$

by appeal to the continuity of $\varphi_{2n}(h_o, 0)$, we deduce that there is at least one h_o^* in $\mathcal{D} < h_o^* < \mathcal{E}$ which satisfies (2.10). In order to use (2.8) and

(2.9) we define ρ by

$$\rho = (A + 1)\omega^{-3}; \tag{2.13}$$

we need to remark that a consequence of our definitions of $k_5(A)$ and $C_4(A)$ is

$$n < \rho(k - AC_4)^3\omega^3. \tag{2.14}$$

Let us now return to (1.2). Our criterion for periodic solutions is the special case of Theorem 1:

THEOREM A. <u>For the solutions of</u> (1.2) <u>satisfying</u> $\dot{x}(0) = 0$, <u>a necessary and sufficient further condition that</u> $x(t)$ <u>should have period</u> $2m\pi$ <u>is</u> $\dot{x}(m\pi) = 0$.

Hence, if we determine an h_0^* for which (2.10) holds, the solution $x(t; h_0^*, 0; 0)$, or $x^*(t)$ say, will have the following properties:

(i) $x^*(t)$ <u>has period</u> $2m\pi$ <u>and</u> $x^*(0) > 0$,

(ii) $\dot{x}^*(0) = 0$, <u>and</u>

(iii) <u>all the stationary points of</u> $x^*(t)$ <u>are positive maxima or negative minima, and it has</u> n <u>of each in</u> $0 \le t < 2m\pi$.

In particular if $n = mk + s$ where, for some A, $k > k_5(A)$ and $0 \le s < m \le Ak^2$

we know that an h_0^* can be found. In (2) we made k, m and s explicit by using the heavier notation $x^*(t|k, m, s)$ instead of $x^*(t)$. However, this notation remains ambiguous unless we prove h_0^* unique and we shall avoid it here. It will be more convenient to write $\{x^*(t|k, m, s)\}$ for the <u>class</u> of solutions of (1.2) having the properties (i) to (iii); with this notation we can say (and this is the content of Theorem 2):

<u>If, for some</u> A, <u>we have</u> $k > k_5(A)$ <u>and</u> $0 \leq s < m \leq Ak^2$ <u>then</u> $\{x^*(t|k, m, s)\}$ <u>is not empty; further, there is at least one member for which</u>

$$(k + sm^{-1} - C_4(A)\cdot mk^{-2})\omega < x^*(0) < (k + sm^{-1} + C_4(A)\cdot mk^{-2})\omega \qquad (2.15)$$

<u>and</u>

$$(k + sm^{-1} - C_4(A)\cdot mk^{-2})^4\omega^4 < \dot{x}^{*2}(t) + x^{*4}(t) < (k + sm^{-1} + C_4(A)\cdot mk^{-2})^4\omega^4. \qquad (2.16)$$

The class $\mathcal{C}$ already mentioned in §1 was defined in (2) as consisting of those members of $\bigcup\{x^*(t|k, m, s)\}$ which for some A had $k > k_5(A)$, $0 \leq s < m \leq Ak^2$ and satisfied (2.15) and (2.16).

In all the above work we have restricted ourselves to solutions having only positive maxima and negative minima, that is to solutions which describe cycles. By rearranging our calculations we can show that this is not too severe a restriction: any periodic solutions (always with $\dot{x}(0) = 0$) whose periods are not too long compared with their amplitudes do describe cycles. We consider an $x^\dagger(t)$ with period $2m\pi$ and $\dot{x}^\dagger(0) = 0$, whose amplitude $x^\dagger(0)$, or $h_o^\dagger$ say, satisfies

$$\omega^{-1} h_o^\dagger > k_5(A) + 1 + AC_4(A) \tag{2.17}$$

$$\omega^{-1} h_o^\dagger > 2 + \frac{5}{2} AC_4(A) \tag{2.18}$$

and
$$\omega^{-1} h_o^\dagger > A^{-\frac{1}{2}} m^{\frac{1}{2}} + 1 + AC_4(A) \tag{2.19}$$

for some positive A. We then find that

h.t. membership

$$x^\dagger(t) \in \{x^*(t|k, m, s)\}$$

for suitable k and s, and in the course of the proof we obtain

$$m < Ak^2 \tag{2.20}$$

and
$$\mathcal{A}(k, m, s; C_4) < x^\dagger(0) < \mathcal{E}(k, m, s; C_4). \tag{2.21}$$

3. <u>The estimation of partial derivatives and the arrangement of the calculations.</u>

In order to describe how rapidly h_{2n} and φ_{2n} vary with h_o and φ_o we shall prove

<u>THEOREM</u> 7. <u>Under the assumption that</u> $p(t)$ <u>has a continuous third derivative, if</u> ρ <u>is a given positive number and</u> $n \leq \rho h_o^3$ <u>then, for large</u> h_o,

$$\frac{\partial(h_{2n}, \varphi_{2n})}{\partial(h_0, \varphi_0)} = \begin{pmatrix} 1 + o(1) & o(h_0^{-1}) \\ -\varpi n h_0^{-2}\{1 + o(1)\} & 1 + o(1) \end{pmatrix},$$

where the tendency to 0 implied in the O-terms is uniform in n.

COROLLARY. There is an $R_2(\rho)$ such that, whenever $h_0 > R_2(\rho)$ and $n \leq \rho h_0^3$, φ_{2n} is defined and

$$\partial\varphi_{2n}/\partial h_0 < 0 .$$

In the enunciation of Theorem 7 we have used the notation

$$\frac{\partial(h_{2n}, \varphi_{2n})}{\partial(h_0, \varphi_0)} = \begin{pmatrix} \partial h_{2n}/\partial h_0 & \partial h_{2n}/\partial\varphi_0 \\ \partial\varphi_{2n}/\partial h_0 & \partial\varphi_{2n}/\partial\varphi_0 \end{pmatrix},$$

and we shall write other Jacobian matrices similarly.

It will be anticipated that the Corollary to Theorem 7 will allow us to show that, in suitably restricted ranges of n and h_0, the h_0^* satisfying (2.10) is unique, and, correspondingly, that when k, m and s are suitably restricted the class $\{x^*(t|k, m, s)\}$ has only one "large" member, which we shall symbolize with $X^*(t|k, m, s)$. When we try to give a clear enunciation of what we have anticipated we have a choice over our use of the parameter ρ

and the corresponding $A = (\rho + 1)\omega^{-3}$ of (2.13). In our estimations and in their application to the existence statement of Theorem 2, carrying these parameters gave some extra generality at very slight cost. When we interest ourselves in uniqueness statements, however, we must choose between working with a fixed value of A or working with a range of A and then eliminating A by taking suitable maxima or minima of the expressions we obtain. The second procedure of course gives a somewhat better result but if we followed it we should have to introduce extra details into our enunciations and proofs which would obscure the method by more than the improved result was worth. We shall give Lemma 17 an enunciation involving A and comment again after that lemma on the possible use of the parameter, but in the remainder of the paper we shall simply put $A = 1$. In particular we shall establish‡:

THEOREM 8. *Suppose that* m *is a positive integer. There exist constants* k_6 *and* C_4 *such that for any integers* k *and* s *satisfying*

$$k > k_6 \quad \text{and} \quad 0 \leq s < m \leq k^2$$

‡I am grateful for suggestions by the referee which have made the enunciation of Theorem 8 much more illuminating than my original form.

there is, under the assumption that $e(t)$ *has a continuous third derivative, one and only one solution of* (1.2), *say* $X^*(t) = X^*(t|k, m, s)$, *for which*

(i) $X^*(t)$ *has period* $2m\pi$,

(ii) $\dot{X}^*(0) = 0$ *and* $X^*(0) > (k - C_4)\omega$ *and*

(iii) *all the stationary points of* $X^*(t)$ *are positive maxima or negative minima and there are* $mk + s$ *of each in* $0 \leq t < 2m\pi$.

This unique solution satisfies

$$(k + sm^{-1} - C_4 mk^{-2})\omega < X^*(0) < (k + sm^{-1} + C_4 mk^{-2})\omega \quad . \tag{3.1}$$

Further, all large periodic solutions of (1.2) *which have* $\dot{x}(0) = 0$ *and have* $x(0)$ *large compared with their periods are of this form: more precisely, if* $x^\dagger(t)$ *has period* $2m\pi$, *if* $\dot{x}^\dagger(0) = 0$,

$$\omega^{-1}x^\dagger(0) - 1 - C_4 > \max\left(k_6, 1 + \tfrac{3}{2} C_4\right) \tag{3.2}$$

and

$$m < (\omega^{-1} x^\dagger(0) - 1 - C_4)^2, \tag{3.3}$$

then there are a k *and an* s *with* $k > k_6$ *and* $0 \leq s < m < k^2$ *such that*

$$x^\dagger(t) = X^*(t|k, m, s).$$

If we compare the enunciation of Theorem 8 with that of Theorem 2, we notice that we now lay heavier requirements on $e(t)$ than the continuous differentiability demanded before, but that the restriction $m \leq Ak^2$ (here of course with $A = 1$) is the same as before. Although it was suggested in (2) that we should have to subject both $e(t)$ and m to heavier conditions, I have since found that there is a choice of hypotheses. Alternative forms of Theorem 7 are briefly considered in §7, and the corresponding forms of Theorem 8 in §8; for Theorems 9 and 10 we adopt the continuity of $\ddot{e}(t)$ outright as our hypothesis.

On the basis of Theorem 7 the proofs of Theorems 8 and 9 are short. For the proof of Theorem 7, we first, in §4, find estimates for the elements of $\partial(h_1, \varphi_1)/\partial(h_0, \varphi_0)$. If h_0 is large $x(t)$, $= x(t;\, h_0,\, 0;\, \varphi_0)$, decreases monotonically to 0 in the range $\varphi_0 \leq t \leq \varphi_1$; hence it has an inverse function which increases as x decreases in the range $h_0 \geq x \geq 0$. If for this function we write $t(x)$ we have $t(h_0) = \varphi_0$, $t(0) = \varphi_1$. From (2.5) we deduce that

[ ] as high as inner ∫ sign

Inner ∫ sign has $\int_{x'}^{h_0}$ as range

$$t(x) = \varphi_0 + \int_x^{h_0} \left[h_0^4 - x'^4 - 2\int_{x'}^{h_0} p\{t(x'')\}\,dx''\right]^{-\frac{1}{2}} dx',$$

i.e. xi

and if we write $x = h_0\xi$ this may be rewritten

Inner ∫ sign has $\int_{\xi'}^{1}$ as range

] as high as inner ∫ sign

$$t(h_0\xi) = \varphi_0 + h_0^{-1}\int_\xi^1 \left[1 - \xi'^4 - 2h_0^{-3}\int_{\xi'}^1 p\{t(h_0\xi'')\}\,d\xi''\right]^{-\frac{1}{2}} d\xi'. \qquad (3.4)$$

The notation $t(x)$ is abbreviated: we should write $t(x;\, h_0,\, \varphi_0)$. This point is relevant since, when we have written $x = h_o\xi$, we shall take the license of writing

$$\frac{\partial t}{\partial h_o} \quad \text{or} \quad \frac{\partial t(h_o\xi)}{\partial h_o} \quad \text{for} \quad \left\{\frac{\partial}{\partial h_o}\, t(h_o\xi;\ h_o,\ \varphi_o)\right\}_{\xi,\varphi_o} .$$

Once we have the estimates of §4 we can write down similar estimates corresponding to other ranges of t. In §7, these are combined (by multiplying Jacobian matrices) to give an estimate for the desired $\underset{\sim}{\partial}(h_{2n},\ \varphi_{2n})/\underset{\sim}{\partial}(h_o,\ \varphi_o)$. In order to effect this combination we need to have estimated (in §5) certain integrals which arise, and to have obtained (in §6) estimates for the products of matrices.

4. <u>The estimation of derivatives of</u> h_1 <u>and</u> φ_1. We know that, for large h_o, both h_1 and φ_1 are differentiable functions of h_o and φ_o. In this section we show that there are simple estimates for their derivatives. As might be expected, the estimates obtained by formal differentiation of (2.6) and (2.7) are valid; with the exception of the estimate for $\partial h_1/\partial\varphi_o$ they are accurate enough for our purposes.

It is to be observed that the integral in (3.4) is improper; to justify differentiation under the integral sign in Lemmas 3 and 4 we establish, by an easy appeal to the principle of dominated convergence,

[illegible] <u>LEMMA</u> 1. <u>If</u> $g(\xi, \lambda)$ <u>has a partial derivative with respect to</u> λ, <u>which is continuous in</u> ξ <u>and</u> λ <u>for</u> $0 \leq \xi \leq 1$ <u>and</u> λ <u>in some range, then for the</u>

same set of (ξ, λ)

$$G(\xi, \lambda) = \int_{\xi}^{1} (1 - \xi')^{-\frac{1}{2}} g(\xi', \lambda) d\xi'$$

has a partial derivative with respect to λ, namely

$$G_{\lambda}(\xi, \lambda) = \int_{\xi}^{1} (1 - \xi')^{-\frac{1}{2}} g_{\lambda}(\xi', \lambda) d\xi' \ .$$

LEMMA 2. For $0 \le \xi \le 1$, $h_o > r$ and all φ_o the function $t(h_o \xi; h_o, \varphi_o)$ has partial derivatives with respect to h_o and φ_o, and these are continuous in ξ, h_o and φ_o.

It is clear that $t(h_o \xi)$ is defined by

$$X \equiv x(t; h_o, 0; \varphi_o) - h_o \xi = 0.$$

When $\xi \neq 1$, $\partial X/\partial t \neq 0$ and our assertion follows by the standard implicit function theorem. Evidently $\partial t/\partial h_o$ is determined by

$$\dot{x}(t; h_o, 0; \varphi_o) \frac{\partial t}{\partial h_o} + \frac{\partial x}{\partial h_o} - \xi = 0, \tag{4.1}$$

and $\partial t/\partial \varphi_o$ is determined similarly.

If $\xi = 1$, then $t = \varphi_o$ for all h_o and φ_o, and therefore has partial derivatives, namely

$$\partial t/\partial h_o = 0 \quad \text{and} \quad \partial t/\partial \varphi_o = 1.$$

To show that these derivatives are continuous as $\xi \to 1$ (necessarily from below), we write $t = \varphi_o + \eta$, which makes $\eta = \underline{o}(1)$, and note first that

$$\dot{x}(t;\ h_o,\ 0;\ \varphi_o) = \{\ \ddot{x}(\varphi_o) + \underline{o}(1)\ \}\ \eta,$$

and, secondly, since $\partial x/\partial h_o$ is the solution of $\ddot{u} + 6x^2(t)u = 0$ for which $u(\varphi_o) = 1$, $\dot{u}(\varphi_o) = 0$, that

raising of l.c. which ok accidental

$$\partial x/\partial h_o = 1 - \{3h_o^2 + \underline{O}(1)\}\eta^2$$

lowering of l.c. which ok accidental

and

$$h_o\xi = h_o + \tfrac{1}{2}\{\ddot{x}(\varphi_o) + \underline{o}(1)\}\eta^2\ .$$

Substitution in (4.1) gives us that $\partial t/\partial h_o \to 0$ as $\eta \to 0$. Similarly $\partial t/\partial \varphi_o \to 1$ as $\eta \to 0$.

<u>LEMMA</u> 3. <u>For large</u> h_o,

$$\frac{\partial h_1}{\partial h_o} = 1 + p(\varphi_o)h_o^{-3} + \underline{O}(h_o^{-4}) \tag{4.2}$$

<u>and</u>

$$\frac{\partial \varphi_1}{\partial h_o} = -\tfrac{1}{2}\,\sigma\, h_o^{-2} + \underline{O}(h_o^{-5}). \tag{4.3}$$

If we rewrite (3.4) as

$$h_o\{t(h_o\xi) - \varphi_o\} = \int_\xi^1 (1 - \xi')^{-\frac{1}{2}}\cdot\{f(\xi'\ ;\ h_o,\ \varphi_o)\}^{-\frac{1}{2}}\, d\xi', \tag{4.4}$$

where

$$f(\xi'; h_o, \varphi_o) = 1 + \xi' + \xi'^2 + \xi'^3 - 2h_o^{-3} \cdot \frac{1}{1-\xi'} \int_{\xi'}^{1} p\{t(h_o\xi'')\} d\xi'', \tag{4.5}$$

it is evident that Lemma 1 can be applied to the improper integral; we obtain

$$h_o \frac{\partial t}{\partial h_o} + \{t(h_o\xi) - \varphi_o\} = -\tfrac{1}{2} \int_{\xi}^{1} (1-\xi')^{-\frac{1}{2}} \cdot f^{-\frac{3}{2}} \cdot \frac{\partial f}{\partial h_o} \cdot d\xi' \tag{4.6}$$

where

$$\frac{\partial f}{\partial h_o} = 6h_o^{-4} \cdot \frac{1}{1-\xi'} \int_{\xi'}^{1} p\{t(h_o\xi'')\}\, d\xi'' -$$

$$-2h_o^{-3} \cdot \frac{1}{1-\xi'} \int_{\xi'}^{1} \dot{p}\{t(h_o\xi)\} \cdot \frac{\partial t}{\partial h_o} \cdot d\xi'' \,. \tag{4.7}$$

Write

$$\mu = \max_{0 \le \xi \le 1} \left| \frac{\partial t}{\partial h_o} \right| ,$$

and suppose that μ is attained when $\xi = \xi_o$. Put $\xi = \xi_o$ in (4.6) and we find

$$h_o\mu + \underline{O}(h_o^{-1}) = \underline{O}(h_o^{-4}) + \underline{O}(\mu h_o^{-3})$$

whence

$$\mu = \underline{O}(h_o^{-2}). \tag{4.8}$$

Now put $\xi = 0$ in (4.6); we obtain

$$h_o \frac{\partial\varphi_1}{\partial h_o} + \varphi_1 - \varphi_o = \underline{O}(h_o^{-4})$$

which gives (4.3).

Finally, by writing $t_o = \varphi_o$ and $t = \varphi_1$ in (2.5) and changing the variable of integration to ξ, we obtain

$$h_1^4 = h_o^4 - 2h_o\int_o^1 p\{t(h_o\xi)\}d\xi \quad .$$

If we differentiate this with respect to h_o and use (2.7) in the form

$$h_1^{-3} = h_o^{-3}\{1 + \tfrac{3}{2}p(\varphi_o)h_o^{-3} + \underline{O}(h_o^{-4})\}$$

we obtain (4.2).

<u>LEMMA</u> 4. <u>For large</u> h_o,

$$\frac{\partial h_1}{\partial\varphi_o} = \tfrac{1}{2}h_o^{-3}\int_{\varphi_o}^{\varphi_1} \dot{p}(t)\dot{x}(t)dt + \underline{O}(h_o^{-5})$$

<u>and</u>

$$\frac{\partial\varphi_1}{\partial\varphi_o} = 1 + \underline{O}(h_o^{-4}).$$

By differentiating (4.4) with respect to φ_o we obtain

$$h_o\left(\frac{\partial t}{\partial\varphi_o} - 1\right) = -\tfrac{1}{2}\int_\xi^1 (1-\xi')^{-\frac{1}{2}} \cdot f^{-\frac{3}{2}} \cdot \frac{\partial f}{\partial\varphi_o} \cdot d\xi',$$

where

$$\frac{\partial r}{\partial \varphi_0} = -2h_0^{-3} \cdot \frac{1}{1-\xi'} \int_{\xi'}^{1} \dot{p}[t(h_0\xi'')] \cdot \frac{\partial t}{\partial \varphi_0} \cdot d\xi'' .$$

If now we write

$$\mu = \max_{0\leq\xi\leq 1} \left| \frac{\partial t}{\partial \varphi_0} - 1 \right| ,$$

we find

$$\mu = \underline{O}\,(h_0^{-4}), \tag{4.9}$$

and the proof proceeds similarly to that of Lemma 3.

5. <u>The estimation of</u> $\int \dot{p}(t)\dot{x}(t)dt$ <u>over the ranges</u> (φ_1, φ_3) <u>and</u> (φ_1, φ_5). In each of the ranges (φ_1, φ_2), (φ_2, φ_3), (φ_3, φ_4) and (φ_4, φ_5), $x(t)$ is monotonic, and it takes the value 0 at one end. It will be convenient to write $t^{(12)}(x)$, $t^{(23)}(x)$, $t^{(34)}(x)$ and $t^{(45)}(x)$ for the inverse functions, that is, we think of writing $t^{(01)}(x)$ for the $t(x)$ of §2 and then generalize this notation. We write

$$\tau(\xi) = \int_{\xi}^{1} (1 - \xi'^4)^{-\frac{1}{2}}\, d\xi' ;$$

it will be observed that $\tau(0) = \frac{1}{2}\varpi$ and $\int_0^1 \tau(\xi)d\xi = \frac{1}{4}\pi$.

<u>LEMMA</u> 5. <u>For large</u> h_0,

$$t^{(12)}\,(-h_2\xi) = \varphi_2 - h_o^{-1}\,\tau(\xi) + \underline{O}(h_o^{-4}),$$

$$t^{(23)}\,(-h_2\xi) = \varphi_2 + h_o^{-1}\,\tau(\xi) + \underline{O}(h_o^{-4}),$$

$$t^{(34)}\,(h_4\,\xi) = \varphi_4 - h_o^{-1}\,\tau(\xi) + \underline{O}(h_o^{-4})$$

and

$$t^{(45)}\,(h_4\,\xi) = \varphi_4 + h_o^{-1}\,\tau(\xi) + \underline{O}(h_o^{-4}).$$

In each range we use the relevant analogue of (3.4).

LEMMA 6. *For large* h_o,

$$\int_{\varphi_1}^{\varphi_3} \dot{p}(t)\dot{x}(t)dt = \underline{o}(h_2)\ .$$

We have

$$\int_{\varphi_1}^{\varphi_3} \dot{p}(t)\dot{x}(t)dt = -h_2\int_o^1 \dot{p}\{\varphi_2-\tau(\xi)h_o^{-1} + \underline{O}(h_o^{-4})\}d\xi + h_2\int_o^1\dot{p}\{\varphi_2 + \tau(\xi)h_o^{-1} + \underline{O}(h_o^{-4})\}d\xi \tag{5.1}$$

$$= -h_2\{\dot{p}(\varphi_2) + \underline{o}\,(1)\} + h_2\{\dot{p}(\varphi_2) + \underline{o}(1)\} = \underline{o}(h_2)\ .$$

LEMMA 7. *If* $p(t)$ *has a continuous second derivative, then*

$$\int_{\varphi_1}^{\varphi_3} \dot{p}(t)\dot{x}(t)dt = \tfrac{1}{2}\,\pi\,\ddot{p}\,(\varphi_2) + \underline{o}\,(1)$$

as $h_o \to \infty$, *uniformly in* φ_2.

We have from (5.1)

$$\int_{\varphi_1}^{\varphi_3} \dot{p}(t)\dot{x}(t)dt = h_2\int_0^1 [\{\dot{p}(\varphi_2) + \ddot{p}(\varphi_2)\tau(\xi)h_o^{-1}\} - \{\dot{p}(\varphi_2) - \ddot{p}(\varphi_2)\tau(\xi)h_o^{-1}\} + \underline{o}(h_o^{-1})]d\xi$$

$$= h_2 \cdot 2\ddot{p}(\varphi_2) \cdot \frac{1}{4}\ \pi h_o^{-1} + \underline{o}(1) = \tfrac{1}{2}\pi\, \ddot{p}\,(\varphi_2) + \underline{o}(1).$$

<u>LEMMA</u> 8. <u>If</u> $u(t)$ <u>is periodic and has a continuous second derivative, if</u> η <u>is a positive variable and</u> η' <u>a variable subject to</u> $0 \leq \eta' \leq \eta$, <u>then</u>

$$u(t_o - \eta - \eta') - u(t_o - \eta + \eta') - u(t_o + \eta - \eta') + u(t_o + \eta + \eta') = 4\,\ddot{u}(t_o)\eta\eta' + \underline{o}(\eta^2)$$

<u>as</u> $\eta \to 0$, <u>uniformly in</u> t_o.

<u>LEMMA</u> 9. <u>If</u> $p(t)$ <u>has a continuous third derivative then</u>

$$\int_{\varphi_1}^{\varphi_5} \dot{p}(t)\dot{x}(t)dt = -\tfrac{1}{2}\varpi\pi\ \dddot{p}(\varphi_3)h_o^{-1} + \underline{o}(h_o^{-1})$$

<u>as</u> $h_o \to \infty$, <u>uniformly in</u> φ_3.

Evidently we have

$$\int_{\varphi_1}^{\varphi_2} \dot{p}(t)\dot{x}(t)dt = -h_2\int_0^1 \dot{p}\{\varphi_3 - \tfrac{1}{2}\varpi h_o^{-1} - \tau(\xi)h_o^{-1} + \underline{O}(h_o^{-4})\}d\xi$$

$$= -h_o\int_0^1 \dot{p}\{\varphi_3 - \tfrac{1}{2}\varpi h_o^{-1} - \tau(\xi)h_o^{-1}\}d\xi + \underline{O}(h_o^{-2}),$$

and similar estimates in the other ranges. From these we obtain, by taking $\eta = \frac{1}{2}\varpi h_0^{-1}$ and $\eta' = \tau(\xi)h_0^{-1}$ in Lemma 8,

$$\int_{\varphi_1}^{\varphi_3} \dot{p}(t)\dot{x}(t)dt = -h_0\int_0^1 4\,\bar{\bar{p}}(\varphi_3)\cdot\tfrac{1}{2}\varpi h_0^{-1}\cdot\tau(\xi)h_0^{-1}d\xi + \underline{o}(h_0^{-1})$$

$$= -\tfrac{1}{2}\varpi\pi\,\bar{\bar{p}}(\varphi_3)h_0^{-1} + \underline{o}(h_0^{-1}) \; .$$

6. Approximations to matrices. In this section we consider some estimations of products of matrices containing a large parameter h. We shall apply this work in §7 by taking h as h_0 or h_{2N}. The matrix

$$\underset{\sim}{c}_\nu = \begin{pmatrix} 1+\underline{O}(h^{-4}) & c_\nu h^{-4} + \underline{o}(h^{-4}) \\ -2\varpi h^{-2} + \underline{o}(h^{-2}) & 1+\underline{O}(h^{-4}) \end{pmatrix} \tag{6.1}$$

is typical of the matrices we shall meet as factors.

When, as in (6.1), we have a sequence of matrices involving error terms depending on h we shall say the estimate is uniform in ν if the constants implied in the $\underline{O}$-terms and the functions tending to 0 implied in the $\underline{o}$-terms can be chosen so as not to depend on ν. We shall say that the matrix $\underset{\sim}{a} = (a_{ij})$ is dominated by $\underset{\sim}{A} = (A_{ij})$ and write

$$\underset{\sim}{a} \ll \underset{\sim}{A}$$

if, for all i, j, $|a_{ij}| \le A_{ij}$.

It will be convenient to use the notation $\prod_{\nu=1}^{n} a_\nu$ for the product with the factors in the order $a_n a_{n-1} \cdots a_1$.

LEMMA 10. If, for all ν, $a_\nu \ll A_\nu$ and $b_\nu \ll B_\nu$ then

$$\prod_{\nu=1}^{n} (a_\nu + b_\nu) - \prod_{\nu=1}^{n} a_\nu \ll \prod_{\nu=1}^{n} (A_\nu + B_\nu) - \prod_{\nu=1}^{n} A_\nu .$$

LEMMA 11. If C is a 2×2 matrix with distinct latent roots λ_1 and λ_2 then $C^N = f_N C + g_N I$, where

$$f_N = (\lambda_1^N - \lambda_2^N)/(\lambda_1 - \lambda_2) \tag{6.2}$$

and

$$g_N = -\lambda_1\lambda_2(\lambda_1^{N-1} - \lambda_2^{N-1})/(\lambda_1 - \lambda_2). \tag{6.3}$$

LEMMA 12. Suppose that c_ν are matrices satisfying (6.1) with the estimate uniform in ν and that $|c_\nu| \leq c$. If K is a constant, then for large h and $N < Kh$

$$\prod_{\nu=1}^{N} c_\nu = \begin{pmatrix} 1 + O(h^{-3}) & \sum_{\nu=1}^{n} c_\nu \cdot h^{-4} + o(h^{-3}) \\ -2\omega N h^{-2}\{1 + o(1)\} & 1 + O(h^{-3}) \end{pmatrix},$$

with the estimate uniform in N.

The hypothesis implies that there are a function $\eta(h)$ tending to 0 and a constant C such that, for all ν,

$$\underset{\sim}{\mathfrak{c}}_\nu - \begin{pmatrix} 1 & c_\nu h^{-4} \\ 0 & 1 \end{pmatrix} \ll \underset{\sim}{C} - \begin{pmatrix} 1 & ch^{-4} \\ 0 & 1 \end{pmatrix} \tag{6.4}$$

and also

$$\underset{\sim}{\mathfrak{s}}_\nu - \begin{pmatrix} 1 & 0 \\ -2\mathcal{S}h^{-2} & 1 \end{pmatrix} \ll \underset{\sim}{C} - \begin{pmatrix} 1 & 0 \\ 2\mathcal{S}h^{-2} & 1 \end{pmatrix}, \tag{6.5}$$

where

$$\underset{\sim}{C} = \begin{pmatrix} 1 + Ch^{-4} & ch^{-4} + c\eta(h)h^{-4} \\ 2\mathcal{S}h^{-2} + 2\mathcal{S}\eta(h)h^{-2} & 1 + Ch^{-4} \end{pmatrix} .$$

By applying Lemma 10 to (6.4) and (6.5) respectively we obtain

$$\prod_{\nu=1}^{N} \underset{\sim}{\mathfrak{c}}_\nu - \begin{pmatrix} 1 & \sum\limits_{\nu=1}^{N} c_\nu \cdot h^{-4} \\ 0 & 1 \end{pmatrix} \ll \underset{\sim}{C}^N - \begin{pmatrix} 1 & cNh^{-4} \\ 0 & 1 \end{pmatrix} \tag{6.6}$$

and

$$\prod_{\nu=1}^{N} \underset{\sim}{\mathfrak{s}}_\nu - \begin{pmatrix} 1 & 0 \\ -2\mathcal{S}Nh^{-2} & 1 \end{pmatrix} \ll \underset{\sim}{C}^N - \begin{pmatrix} 1 & 0 \\ 2\mathcal{S}Nh^{-2} & 1 \end{pmatrix} . \tag{6.7}$$

The latent roots of $\underset{\sim}{C}$ are

$$\lambda_1 = 1 + \Delta + Ch^{-4} \qquad \text{and} \qquad \lambda_2 = 1 - \Delta + Ch^{-4},$$

where $\Delta \sim \sqrt{(2\varpi c)} \cdot h^{-3}$ as $h \to \infty$. We see that

$$\lambda_1^N = 1 + N\Delta + CNh^{-4} + \tfrac{1}{2}N^2\Delta^2 + \underline{O}(h^{-5})$$

and
$$\lambda_2^N = 1 - N\Delta + CNh^{-4} + \tfrac{1}{2}N^2\Delta^2 + \underline{O}(h^{-5}),$$

and hence, in the notation of Lemma 11,

$$f_N = \{2N\Delta + \underline{O}(h^{-5})\}/(2\Delta) = N + \underline{O}(h^{-2})$$

and
$$\begin{aligned} f_N + g_N &= (1 - \lambda_2)f_N + \lambda_2^N \\ &= \{N\Delta - CNh^{-4} + \underline{O}(h^{-5})\} + \{1 - N\Delta + CNh^{-4} + \underline{O}(h^{-4})\} \\ &= 1 + \underline{O}(h^{-4}). \end{aligned}$$

It follows that the diagonal elements of $\underset{\sim}{C}^N$ are each

$$(f_N + g_N) \cdot 1 + f_N \cdot Ch^{-4} = 1 + \underline{O}(h^{-3}),$$

and that the elements off the diagonal are

$$f_N\{ch^{-4} + c\eta(h)h^{-4}\} = cNh^{-4} + \underline{o}(h^{-3})$$

and $$f_N\{2\varpi h^{-2} + 2\varpi\eta(h)h^{-2}\} = 2\varpi N h^{-2}\{1 + \underline{o}(1)\} .$$

Our assertion now follows from (6.6) and (6.7).

LEMMA 13. _Suppose that_ $N = N(h) < Kh$, _where_ K _is a constant, and that_

$$\underset{\sim}{p}_\kappa = \begin{pmatrix} 1 + \underline{O}(h^{-3}) & \underline{o}(h^{-3}) \\ -2\varpi N h^{-2}\{1 + \underline{o}(1)\} & 1 + \underline{O}(h^{-3}) \end{pmatrix} ,$$

with the estimate uniform in κ. _If_ L _is a constant, then, for large_ h _and any_ q _satisfying_ $q < Lh^2$,

$$\prod_{\kappa=0}^{q-1} \underset{\sim}{p}_\kappa = \begin{pmatrix} 1 + \underline{o}(1) & \underline{o}(h)^{-1} \\ -2\varpi N q h^{-2}\{1 + \underline{o}(1)\} & 1 + \underline{o}(1) \end{pmatrix} ,$$

with the estimate uniform in q.

The estimate for $\underset{\sim}{p}_\kappa$ implies that there are a function $\eta(h)$ tending to 0 and a constant C such that

$$\underset{\sim}{p}_\kappa - \begin{pmatrix} 1 & 0 \\ -2\varpi N h^{-2} & 1 \end{pmatrix} \ll \underset{\sim}{P} - \begin{pmatrix} 1 & 0 \\ 2\varpi N h^{-2} & 1 \end{pmatrix} , \tag{6.8}$$

where

$$\underset{\sim}{P} = \begin{pmatrix} 1 + Ch^{-3} & \eta(h)h^{-3} \\ 2\varpi Nh^{-2}\{1 + \eta(h)\} & 1 + Ch^{-3} \end{pmatrix} .$$

By applying Lemma 10 to (6.8) we obtain

$$\prod_{\kappa=0}^{q-1} \underset{\sim}{P}_{\kappa} - \begin{pmatrix} 1 & 0 \\ -2\varpi Nqh^{-2} & 1 \end{pmatrix} \ll \underset{\sim}{P}^{q} - \begin{pmatrix} 1 & 0 \\ 2\varpi Nqh^{-2} & 1 \end{pmatrix} . \tag{6.9}$$

The latent roots of $\underset{\sim}{P}$ are

$$\lambda_1 = 1 + \Delta + Ch^{-3} \quad \text{and} \quad \lambda_2 = 1 - \Delta + Ch^{-3} ,$$

where $\Delta = \underline{o}(h^{-2})$. We see that

$$\lambda_1^q = (1 + Ch^{-3})^q + q(1 + Ch^{-3})^{q-1}\Delta + \underline{O}(q^2\Delta^2)$$

and
$$\lambda_2^q = (1 + Ch^{-3})^q - q(1 + Ch^{-3})^{q-1}\Delta + \underline{O}(q^2\Delta^2)$$

and hence, in the notation of Lemma 11,

$$\begin{aligned} r_q &= \{2q(1 + Ch^{-3})^{q-1}\Delta + \underline{O}(q^2\Delta^2)\}/(2\Delta) \\ &= q + \underline{O}(qh^{-1}) + \underline{O}(q^2\Delta) = q + \underline{o}(q) \end{aligned}$$

and

$$f_q + g_q = (1 - \lambda_2)f_q + \lambda_2^q$$

$$= \{q\Delta - Cqh^{-3} + \underline{o}(qh^{-2})\} + \{1 - q\Delta + Cqh^{-3} + \underline{o}(q^2h^{-4})\}$$

$$= 1 + \underline{o}(1).$$

These results allow us to estimate the elements of $\underset{\sim}{P}^q$ and our assertion then follows from (6.9).

In the final stage of the proof of Theorem 7 a number of matrices of the form

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$$\underset{\sim}{\sigma}(H) = \begin{pmatrix} 1 + \underline{o}(1) & \underline{o}(h^{-1}) \\ -Hh^{-2}\{1 + \underline{o}(1)\} & 1 + \underline{o}(1) \end{pmatrix} \tag{6.10}$$

will occur. We note that, in this notation, Lemma 13 asserts that $\prod \underset{\sim}{P}_K = \underset{\sim}{\sigma}(2\sigma Nq)$. For these matrices we shall require

<u>LEMMA</u> 14. <u>If</u> K <u>is a constant and</u> $0 < H_i < Kh^3$ <u>for</u> $i = 1, 2,$ <u>then</u>

$$\underset{\sim}{\sigma}(H_1) \cdot \underset{\sim}{\sigma}(H_2) = \underset{\sim}{\sigma}(H_1 + H_2)$$

<u>as</u> $h \to \infty$.

7. <u>The combination of estimates.</u> Whenever the derivatives are defined we shall write

$$\underset{\sim}{m}_i = \frac{\partial(h_i, \varphi_i)}{\partial(h_{i-1}, \varphi_{i-1})} \quad \text{and} \quad \underset{\sim}{M}_i = \frac{\partial(h_i, \varphi_i)}{\partial(h_o, \varphi_o)} \; ;$$

it is evident that the results of Lemmas 3 and 4 may be rewritten

$$\underset{\sim}{m}_1 = \begin{pmatrix} 1 + p(\varphi_o)h_o^{-3} + \underline{O}(h_o^{-4}) & \frac{1}{2}h_o^{-3}\int_{\varphi_o}^{\varphi_1} \dot{p}(t)\dot{x}(t)dt + \underline{O}(h_o^{-5}) \\ -\frac{1}{2}\sigma h_o^{-2} + \underline{O}(h_o^{-5}) & 1 + \underline{O}(h_o^{-4}) \end{pmatrix}.$$

We need to estimate the elements of

$$\underset{\sim}{M}_{2n} = \prod_{\nu=1}^{2n} \underset{\sim}{m}_\nu \; .$$

When the matrices on the right are multiplied together, there is approximate cancellation of the contributions to the element in (1, 2) position of the product. In order to take advantage of this we group the factors and define matrices $\underset{\sim}{c}_\nu$ and $\underset{\sim}{p}_\kappa$ which we shall identify with those considered in §6. First we take the factors (apart from the end ones) four at a time and write

$$\underset{\sim}{c}_\nu = \underset{\sim}{m}_{4\nu+1}\, \underset{\sim}{m}_{4\nu}\, \underset{\sim}{m}_{4\nu-1}\, \underset{\sim}{m}_{4\nu-2} \;\; ;$$

we then take the $\underset{\sim}{c}_\nu$ in groups of N, where $N = [h_o/\omega]$, and write

$$\mathbf{p}_\kappa = \prod_{\lambda=1}^{N} \mathbf{c}_{N\kappa+\lambda} \; .$$

i.e. italic ll

When n is even, we write $\frac{1}{2}n - 1 = Nq + \ell$ with $0 \leq q,\ 0 \leq \ell < N$ and obtain

$$\mathbf{M}_{2n} = \mathbf{m}_{2n}\,\mathbf{m}_{2n-1}\,\mathbf{m}_{2n-2} \cdot \prod_{\lambda=1}^{\ell} \mathbf{c}_{Nq+\lambda} \cdot \prod_{\kappa=0}^{q-1} \mathbf{p}_\kappa \cdot \mathbf{m}_1 \; . \tag{7.1}$$

When n is odd we write $\frac{1}{2}(n-1) = Nq + \ell$ with $0 \leq q,\ 0 \leq \ell < N$ and obtain

$$\mathbf{M}_{2n} = \mathbf{m}_{2n} \cdot \prod_{\lambda=1}^{\ell} \mathbf{c}_{Nq+\lambda} \cdot \prod_{\kappa=0}^{q-1} \mathbf{p}_\kappa \cdot \mathbf{m}_1 \; .$$

As is suggested by the notation, the factor matrices grouped in $\mathbf{p}_\kappa$ correspond approximately to a period of $p(t)$. Similarly, the notation $\mathbf{c}_\nu$ suggests a cycle; it will be observed, however, that $\mathbf{c}_\nu$ corresponds to the interval $\varphi_{4\nu-3} \leq t \leq \varphi_{4\nu+1}$, not $\varphi_{4\nu-4} \leq t \leq \varphi_{4\nu}$ which we have defined as the νth cycle.

<u>LEMMA</u> 15. <u>For large</u> h_o,

$$\mathbf{m}_3\,\mathbf{m}_2 = \begin{pmatrix} 1 + \underline{O}(h_o^{-4}) & \frac{1}{2}h_o^{-3}\int_{\varphi_1}^{\varphi_3} \mathbf{p}(t)\mathbf{x}(t)\,dt + \underline{O}(h_o^{-5}) \\ -\varepsilon h_o^{-2} + \underline{O}(h_o^{-5}) & 1 + \underline{O}(h_o^{-4}) \end{pmatrix} \tag{7.2}$$

<u>and</u>

$$\underset{\sim}{m}_5\, \underset{\sim}{m}_4 = \begin{pmatrix} 1 + \underline{O}(h_o^{-4}) & \frac{1}{2}h_o^{-3} \int_{\varphi_3}^{\varphi_5} p(t)\dot{x}(t)dt + \underline{O}(h_o^{-5}) \\ -\varpi h_o^{-2} + \underline{O}(h_o^{-5}) & 1 + \underline{O}(h_o^{-4}) \end{pmatrix}. \qquad (7.3)$$

Evidently the method of Lemmas 3 and 4 is applicable to the estimation of the elements of $\underset{\sim}{m}_2^{-1} = \partial(h_1, \varphi_1)/\partial(h_2, \varphi_2)$; we need not repeat the work here but can check the signs of coefficients by formally differentiating (6.9) and (6.10) of ($\underset{\sim}{2}$). We obtain, by inverting this matrix, the estimate

$$\underset{\sim}{m}_2 = \begin{pmatrix} 1 + p(\varphi_2)h_2^{-3} + \underline{O}(h_2^{-4}) & \frac{1}{2}h_2^{-3} \int_{\varphi_1}^{\varphi_2} p(t)\dot{x}(t)dt + \underline{O}(h_2^{-5}) \\ -\frac{1}{2}\varpi h_2^{-2} + \underline{O}(h_2^{-5}) & 1 + \underline{O}(h_2^{-4}) \end{pmatrix}.$$

Similarly, from (6.11) and (6.12) of ($\underset{\sim}{2}$), we can check that

$$\underset{\sim}{m}_3 = \begin{pmatrix} 1 - p(\varphi_2)h_2^{-3} + \underline{O}(h_2^{-4}) & \frac{1}{2}h_2^{-3} \int_{\varphi_2}^{\varphi_3} p(t)\dot{x}(t)dt + \underline{O}(h_2^{-5}) \\ -\frac{1}{2}\varpi h_2^{-2} + \underline{O}(h_2^{-5}) & 1 + \underline{O}(h_2^{-4}) \end{pmatrix}.$$

Direct multiplication and use of the estimates $h_2^{-2} = h_o^{-2} + \underline{O}(h_o^{-5})$, $h_2^{-3} = h_o^{-3} + \underline{O}(h_o^{-6})$ now gives (7.2), and (7.3) is established similarly.

<u>LEMMA</u> 16. <u>If</u> η <u>is a positive variable and</u> $u(t)$ <u>is a continuous function with period</u> 2π <u>for which</u>

$\int_0^{2\pi} u(t)dt = 0$ then, if N is defined as the greatest integer in $2\pi/\eta$,

$$\eta \sum_{\nu=0}^{N-1} u(t_o + \nu\eta)$$

tends to 0 as $\eta \to 0$, uniformly in t_o.

Proof of Theorem 7. We recall that we are now assuming $\ddot{p}(t)$ continuous. First, multiply the matrices on the right-hand sides of (7.2) and (7.3), and then apply Lemma 9 to estimate the integral occurring in (1, 2) position. Secondly, replace the error term $\underline{O}(h_o^{-5})$ in (2, 1) position (whose accuracy cannot be fully exploited) by the coarser $\underline{o}(h_o^{-2})$; we obtain

dots on p

$$\mathfrak{L}_1 = \mathfrak{m}_5\mathfrak{m}_4 \cdot \mathfrak{m}_3\mathfrak{m}_2 = \begin{pmatrix} 1 + \underline{O}(h_o^{-4}) & -\frac{1}{4}\varpi\pi\ddot{p}(\varphi_3)h_o^{-4} + \underline{o}(h_o^{-4}) \\ -2\varpi h_o^{-2} + \underline{o}(h_o^{-2}) & 1 + \underline{O}(h_o^{-4}) \end{pmatrix}.$$

In this estimate write $h_{4\nu-4}$ for h_o and $\varphi_{4\nu-1}$ for φ_3; we obtain

$$\mathfrak{L}_\nu = \begin{pmatrix} 1 + \underline{O}(h_{4\nu-4}^{-4}) & -\frac{1}{4}\varpi\pi\ddot{p}(\varphi_{4\nu-1})h_{4\nu-4}^{-4} + \underline{o}(h_{4\nu-4}^{-4}) \\ -2\varpi h_{4\nu-4}^{-2} + \underline{o}(h_{4\nu-4}^{-2}) & 1 + \underline{O}(h_{4\nu-4}^{-4}) \end{pmatrix}, \tag{7.4}$$

a result which we rearrange in order to discuss $\underset{\sim}{p}_K$.

If ν varies in any way subject to $\nu \leq \frac{1}{2}n \leq \frac{1}{2}\rho h_o^3$, the arguments leading to (2.8) need only slight modification to give

$$\varphi_{4\nu-1} = \varphi_3 + 2\varpi(\nu - 1)h_0^{-1} + \underline{O}(\nu^2 h_0^{-5}) + \underline{O}(\nu h_0^{-4}) \,. \tag{7.5}$$

When ν lies in the range $1 \leq \nu \leq N$, the second error term of (7.5) is the more important and we obtain

$$\varphi_{4\nu-1} = \varphi_3 + 2\varpi(\nu-1)h_0^{-1} + \underline{O}(h_0^{-3}), \tag{7.6}$$

which gives, since $\dddot{p}(t)$ is continuous, that $\dddot{p}(\varphi_{4\nu-1}) = \dddot{p}\{\varphi_3 + 2\varpi(\nu-1)h_0^{-1}\} + \underline{o}(1)$; if this last is used in (7.4) and $h_{4\nu-4}$ everywhere replaced by h_0 and suitable error terms we are led to

$$\underset{\sim}{c}_\nu = \begin{pmatrix} 1 + \underline{O}(h_0^{-4}) & -\frac{1}{4}\varpi\pi\,\dddot{p}\{\varphi_3 + 2\varpi(\nu-1)h_0^{-1}\}h_0^{-4} + \underline{o}(h_0^{-4}) \\ -2\varpi h_0^{-2} + \underline{o}(h_0^{-2}) & 1 + \underline{O}(h_0^{-4}) \end{pmatrix} \,. \tag{7.7}$$

If we write $u(t) = \dddot{p}(t)$, $\eta = 2\varpi h_0^{-1}$ and $c_\nu = -\frac{1}{4}\varpi\pi\dddot{p}\{\varphi_3 + 2\varpi(\nu-1)h_0^{-1}\}$ we obtain from Lemma 16

$$\sum_{\nu=1}^{N} c_\nu = \underline{o}(h_0). \tag{7.8}$$

If, further, we write $c = \frac{1}{4}\varpi\pi \max |\dddot{p}(t)|$ and identify h_0 with the h of Lemma 12 we find that, at least for $\kappa = 0$,

$$\underset{\sim}{P}_\kappa = \begin{pmatrix} 1 + \underline{O}(h_0^{-3}) & \underline{o}(h_0^{-3}) \\ -2\varpi N h_0^{-2}\{1 + \underline{o}(1)\} & 1 + \underline{O}(h_0^{-3}) \end{pmatrix} \,. \tag{7.9}$$

When $\nu > N$, we write $\nu = N\kappa + \lambda$, where $1 \leq \lambda \leq N$, and rewrite (7.6) as

$$\varphi_{4\nu-1} = \varphi_{4N\kappa+3} + 2\varpi(\lambda-1)h_{4N\kappa}^{-1} + \underline{O}(h_{4N\kappa}^{-3});$$

we deduce that

$$\varphi_{4\nu-1} = \varphi_{4N\kappa+3} + 2\varpi(\lambda-1)h_o^{-1} + \underline{O}(h_o^{-2})$$

uniformly in κ, and, by using this instead of (7.6), find that the estimate (7.9) is valid for $\kappa = 1, 2, \ldots, q-1$ as well as for $\kappa = 0$, and uniformly in κ.

A similar argument gives

$$\prod_{\lambda=1}^{l} \mathfrak{c}_{Nq+\lambda} = \begin{pmatrix} 1 + \underline{O}(h_o^{-3}) & \underline{O}(h_o^{-3}) \\ -2\varpi l h_o^{-2}\{1 + \underline{o}(1)\} & 1 + \underline{O}(h_o^{-3}) \end{pmatrix},$$

the only difference from the argument leading to (7.9) being that, since no analogue of (7.8) is available, the error term in (1, 2) position cannot be reduced from $\underline{O}(h_o^{-3})$ to $\underline{o}(h_o^{-3})$. This is of no importance because it is sufficient to use the coarsened estimate conveniently written, in the notation of (6.10) as

$$\prod_{\lambda=1}^{l} \mathfrak{c}_{Nq+\lambda} = \mathfrak{g}\,(2\varpi l). \qquad (7.10)$$

Whether n is odd or even we have $N_q \leq N_q + l < \frac{1}{2}n$, and therefore $q < \frac{1}{2}n\varpi/(h_o - \varpi) = \underline{O}(h_o^2)$. Evidently the hypotheses of Lemma 13 are fulfilled and we obtain

$$\prod_{\kappa=0}^{q-1} p_\kappa = g(2\vartheta Nq). \tag{7.11}$$

Finally, for n even, it is clear that we have

$$m_{2n} m_{2n-1} m_{2n-2} = g(\tfrac{3}{2}\vartheta) \tag{7.12}$$

and

$$m_1 = g(\tfrac{1}{2}\vartheta), \tag{7.13}$$

and that if we substitute from (7.12), (7.10), (7.11) and (7.13) in the right-hand side of (7.1) Lemma 14 can be applied repeatedly. If n is odd, we proceed similarly; in either case we obtain $M_{2n} = g(\vartheta n)$, which is the assertion of Theorem 7.

Remark on alternative hypotheses. If we vary the hypotheses on $p(t)$ we obtain better or worse estimates for the elements of c_ν. The critical element is that in (1, 2) position, since it is the product of this and $2\vartheta h_o^{-2}$ which determines the size of $\lambda_1 - \lambda_2$ for C. Under the assumption that $p(t)$ had a continuous third derivative we obtained (7.7) whose critical element was $c_\nu h_o^{-4} + o(h_o^{-4})$. We might say that this element was effectively $o(h_o^{-4})$ because, on account of (7.8), the sum of the terms $c_\nu h_o^{-4}$ contributed $o(h_o^{-3})$ instead of $O(h_o^{-3})$ to p_κ and we obtained the same final estimate for M_{2n} as if we had started with a term $o(h_o^{-4})$ in c_ν.

If we assume about $p(t)$, as in (2) and (3), only that it has a continuous derivative, Lemma 6 shows that we obtain an estimate for $\underset{\sim}{c}_{\nu}$ with a critical element $\underline{o}(h_o^{-2})$. Working from this estimate we can obtain, if we impose the restriction $n < \rho' h_o^2$, an estimate for $\underset{\sim}{M}_{2n}$ similar to that of Theorem 7; here again we are led to the corollary that, for large h_o, $\partial\varphi_{2n}/\partial h_o < 0$.

If we assume that $p(t)$ has a continuous second derivative Lemma 7 shows that we obtain an estimate for $\underset{\sim}{c}_{\nu}$ whose critical element is <u>effectively</u> $\underline{o}(h_o^{-3})$.[5] We can now carry through similar work if we impose the restriction $n < \rho'' h_o^2$.

8. <u>The uniqueness of</u> $X^*(t|k, m, s)$. We now return to (1.2) with an even $e(t)$ and refine the results of §2 by exploiting the Corollary to Theorem 7.

<u>LEMMA</u> 17. <u>Corresponding to an assigned positive</u> A <u>there is a</u> $k_6(A)$ <u>which, under the assumption that</u> $e(t)$ <u>has a continuous third derivative, has the following properties</u>: <u>if</u> k, m <u>and</u> s <u>are integers satisfying</u> $k > k_6(A)$ <u>and</u> $0 \leq s < m \leq Ak^2$ <u>there is one and only one member of</u> $\{x^*(t|k, m, s)\}$ <u>for which</u>

$$x^*(0) > (k - AC_4(A)) \cdot \omega, \tag{8.1}$$

<u>say</u> $X^*_A(t|k, m, s)$. <u>Further</u>, $X^*_A(t|k, m, s)$ <u>satisfies</u> (2.15).

Define $k_6(A) = \max[k_5(A), \omega^{-1} R_2(\rho) + AC_4(A)]$ where, as in (2.13), $\rho = (A + 1)\omega^{-3}$.

Since $k_6(A) \geq k_5(A)$ we know that when $k > k_6(A)$ the class $\{x^*(t|k, m, s)\}$ is not empty and that at least one member satisfies (2.15), and a fortiori (8.1). To see that this member is unique, we note that, if $h_o = x^*(0)$ satisfies (8.1) and $n = mk + s$, we have on the one hand

$$h_o > (k_6 - AC_4(A)) \cdot \omega \geq R_2(\rho)$$

and on the other, by use of (2.14),

$$n < \rho(k - AC_4 (A))^3\omega^3 < \rho h_o^3.$$

These inequalities show that $\varphi_{2n}(h_o, 0)$ decreases and that the equation $\varphi_{2n}(h_o, 0) = m\pi$ which has at least one solution has just one.

Remarks. (i) As mentioned in §3, we shall in the sequel simplify our work by choosing $A = 1$; we may then treat $k_6 = k_6(1)$ and $C_4 = C_4(1)$ as constants. If we choose the alternative course of carrying A as a parameter and taking maxima and minima our work is much simpler if we can assert that $k_6(A)$, and other functions, have inverses. On this account it is convenient to assume that $C_4(A)$, $k_5(A)$ and $R_2(\rho)$ are continuous and (strictly) increasing in their arguments. To make this assumption involves no loss of generality: we need only observe that the essential properties required of these functions when we introduced them were that they should satisfy certain one-sided inequalities, and that we can, if necessary, re-define them to be continuous and increasing without losing these properties.

(ii) It will be clear that, corresponding to the modifications of Theorem 7 mentioned in §7 we can obtain modified versions of Lemma 17 (and then of Theorem 8, again with a choice over our treatment of the parameter). When we know only that $e(t)$ has a continuous derivative we assume $m \leq A'k$; when we know that $e(t)$ has a continuous second derivative we assume $m \leq A''k^2$.

In the remaining work we shall require that $e(t)$ has a continuous third derivative.

<u>Proof of Theorem 8</u>. The existence and uniqueness of $X^*(t|k, m, s)$ for given k, m, s follows from Lemma 17 if we take $A = 1$.

If the solution $x^\dagger(t)$ satisfies (3.2) and (3.3) it is clear that $x^\dagger(t)$ satisfies (2.17), (2.18) and (2.19) with $A = 1$, which implies that there are a k and an s for which

not membership

$$x^\dagger(t) \in \{x^*(t|k, m, s)\}$$

and (2.20), that is, $m < k^2$, hold. If we slightly weaken (2.21) and rearrange it we obtain

$$\omega^{-1} x^\dagger(0) + C_4 > k > \omega^{-1} x^\dagger(0) - 1 - C_4$$

and hence we have $k > k_6$ and $x^{\dagger}(0) > (k - C_4)\omega$ from (3.2). Lemma 17 now gives $x^{\dagger}(t) = X^*(t|k, m, s)$.

COROLLARY 1. *If* $k \geq k_6$, $0 \leq s < m \leq k^2$ *and, for a positive integer* γ, $m = \gamma m'$ *and* $s = \gamma s'$, *then*

$$X^*(t|k, m, s) = X^*(t|k, m', s').$$

COROLLARY 2. *If* $P(\alpha, 0)$ *is a fixed point of* T *of order* m *on the* a*-axis and if*

$$\alpha > \omega \max (k_6 + 1 + C_4, \ 2 + \tfrac{5}{2} C_4) \tag{8.2}$$

and

$$m < (\omega^{-1} \alpha - 1 - C_4)^2 \tag{8.3}$$

there are a k *and an* s *with* $k > k_6$ *and* $0 \leq s < m < k^2$ *such that*

$$\alpha = X^*(0|k, m, s).$$

Proof of Theorem 9. We need only rename $X^*(t|k, 1, 0)$ as $x_k(t)$ to obtain our enunciated form from Theorem 8.

9. The arrangement of fixed points. In this section we shall combine our uniqueness results with the work of (3) so as to obtain Theorem 10. In (3)

we began from the result that, <u>when</u>[‡] $k \geq p$, (1.2) <u>has at least one solution</u> $x_k(t)$ <u>for which</u>

(i) $x_k(0) > 0$, $\dot{x}_k(0) = 0$ <u>and</u> $x_k(t)$ <u>has period</u> 2π,

(ii) $x_k(t) = x_k(2\pi - t)$,

(iii) $x_k(t)$ <u>has, in</u> $0 \leq t < 2\pi$, k <u>positive maxima,</u> k <u>negative minima and no other stationary points, and</u>

(iv) $(k - \frac{1}{2})^4\omega^4 < \dot{x}_k^2(t) + x_k^4(t) < (k + \frac{1}{2})^4\omega^4$.

This was introduced as a special case of Theorem 2 and of course did not use Theorem 9 by which it is superseded for large k; we simply took $x_k(t)$ to be a member of $\{x^*(t|k, 1, 0)\}$ which satisfied (iv). Having chosen a particular $x_k(t)$ we wrote a_k for $x_k(0)$ and F_k for the point $(a_k, 0)$ or the (a, b)-plane. We continued by showing that (1.2) has periodic solutions for which the difference $x(t) - x_p(t)$ has simple properties.

Theorems 5 and 6 are stated in terms of fixed points. Here rather than Theorem 5 we need to quote its Corollary that <u>if the integers</u> k, m <u>and</u> s <u>satisfy</u>

$$k > p \quad \underline{\text{and}} \quad 0 \leq s < m$$

[‡]Here p is a constant and has no connection with the $p(t)$ of the early sections of this paper. This is a suitable place to recall that some features of the notation of (2) were modified in (3).

there is at least one solution $x^{**}(t)$ *of* (1.2) *with the properties:*

(i) $x^{**}(t)$ *has period* $2m\pi$,

(ii) $\dot{x}^{**}(0) = 0$,

(iii) *all the stationary points of* $x^{**}(t) - x_p(t)$ *are positive maxima or negative minima and this difference has* $mk + s$ *of each in* $0 \leq t < 2m\pi$, *and*

(iv) $a_k \leq x^{**}(0) \leq a_{k+1}$.

We shall write $\{x^{**}(t|k, m, s)\}$ for the class of solutions having these properties. In (3) we wrote $x^{**}(t|k, m, s)$ ambiguously for such a solution and were not inconvenienced since all our emphasis was on existence problems but we shall avoid it here. When we know that $\{x^{**}(t|k, m, s)\}$ has only one member we shall write $X^{**}(t|k, m, s)$ for it.

If, for fixed k and m, we consider all s less than m and prime to it, the points of abscissae $\{x^{**}(0|k, m, s)\}$ are fixed points of order m lying on F_kF_{k+1}. Evidently if we let m vary we obtain for each m at least $\varphi(m)$ fixed points of order m; whether or not there are exactly $\varphi(m)$ for each m, we showed (as Theorem 6) that *it is possible to choose a fixed point* P_{sm} *to correspond to each irreducible fraction* s/m *with* $0 \leq s \leq m$ *so that* P_{sm} *is of order* m *and that if* $s/m < s'/m'$ *then* P_{sm} *lies to the left of* $P_{s'm'}$. In the course of the proof we showed that *if* $x(t)$ *is the solution corresponding to* P_{sm} *then* $x(t) - x_p(t)$ *has* $mk + s$ *positive maxima in* $0 \leq t < 2m\pi$. We note that by allowing equality of s and m in $0 \leq s \leq m$ we provide for the definition of P_{01} and P_{11} (which must evidently be F_k and F_{k+1} respectively if these points are uniquely determined).

In addition to these theorems we need to recall from (3) the idea that, when we consider a function $u(t)$ which

(i) has only positive maxima and negative minima in the interval $[0, \lambda]$

(ii) has $u(\lambda - t) = u(t)$ and

(iii) has a maximum at $t = 0$,

we can define a function $\theta(t)$ with $\theta(0) = 0$ which measures the rotation of the curve

$$\Gamma: \ \xi = u(t), \quad \eta = \dot{u}(t), \quad \zeta = t,$$

about the line $(0, 0, t)$. This function is connected with the number, say 2γ, of stationary points of $u(t)$ in $[0, \lambda)$ by

$$\theta(\lambda) = 2\theta(\tfrac{1}{2}\lambda) = 2\pi\gamma \ . \tag{9.1}$$

Further, if $v(t)$ is a given function we can under similar conditions define $\theta_v(t)$ to measure the rotation of Γ about the curve $(v(t), \dot{v}(t), t)$. If $u(t) - v(t)$ has a positive maximum at $t = 0$ and $v(\lambda - t) = v(t)$, then under the conditions

$$\dot{u}^2(t) + u^4(t) > K, \tag{9.2}$$

$\dot{v}^2$

and

$$\dot{v}^2(t) + v^4(t) < K, \tag{9.3}$$

where K is a constant, we can compare θ and θ_v and show that $\theta_v(\frac{1}{2}\lambda) = \theta(\frac{1}{2}\lambda)$. In our applications to (1.2) we take $u(t)$ to be a solution of the equation and $v(t)$ to be the special solution $x_p(t)$.

Finally we need to mention explicitly that, as appeared in the proof of Theorem A, <u>if</u> $x(t)$ <u>is a solution of</u> (1.2) <u>with period</u> $2m\pi$ <u>for which</u> $\dot{x}(0) = 0$, <u>then</u> $x(t) = x(2m\pi - t)$.

<u>LEMMA</u> 18. <u>If</u>

$$k > \max\left(p + 2 + \frac{5}{2} C_4,\ k_6 + \frac{3}{2} + C_4,\ \frac{5}{2} + \frac{5}{2} C_4\right)$$

<u>and</u>

$$0 \leq s < m \leq \left(k - \frac{3}{2} - C_4\right)^2$$

<u>then</u> $\{x^{**}(t|k, m, s)\}$ <u>has just one member, namely</u> $X^{**}(t|k, m, s)$, <u>and</u>

$$X^{**}(t|k, m, s) = X^{*}(t|k, m, s).$$

Since $k > p$, $\{x^{**}(t|k, m, s)\}$ is not empty. If $x^{\dagger}(t)$ denotes any (fixed) member of this class we have

$$x^{\dagger}(0) \geq a_k > (k - \tfrac{1}{2})\omega$$

and therefore our hypotheses imply (3.2) and (3.3) of Theorem 8. It follows that there are a k' and an s' such that

$$x^{\dagger}(t) = X^*(t|k', m, s'), \tag{9.4}$$

and this gives

$$k' > k - \frac{3}{2} - C_4,$$

since, by (3.1), $(k' + 1 + C_4)\omega > x^{\dagger}(0) > (k - \frac{1}{2})\omega$.

We note also that (2.16) with $A = 1$ gives

$$\dot{x}^{\dagger 2}(t) + x^{\dagger 4}(t) > (k' - \frac{3}{2} C_4)^4\omega^4 .$$

Write $u(t) = x^{\dagger}(t)$ and $\lambda = 2m\pi$, then (9.4) shows that the conditions required for (9.1) are satisfied, and hence

$$\theta(m\pi) = (mk' + s')\pi.$$

Write $v(t) = x_p(t)$ then the defining properties of $\{x^{**}(t|k, m, s)\}$ give us similarly

$$\theta_v(m\pi) = (mk + s)\pi .$$

If we write $K = (p + \frac{1}{2})^4\omega^4$, we see that

$$\dot{u}^2(t) + u^4(t) \geq (k' - \frac{3}{2} C_4)^4\omega^4 > (k - \frac{3}{2} - \frac{5}{2} C_4)^4\omega^4 > K,$$

that is, that (9.2) holds. Since, evidently, (9.3) holds, we have

$\theta(m\pi) = \theta_v(m\pi)$ or $mk' + s' = mk + s$; this implies, by the use of $0 \leq s' < m$ and $0 \leq s < m$ that $k' = k$ and $s' = s$, and hence (9.4) gives us that $x^\dagger(t)$ is the only member of $\{x^{**}(t|k, m, s)\}$ and our result follows.

Proof of Theorem 10. Write C_5 for the integer which satisfies

$$C_5 \leq 4 + 2C_4 < C_5 + 1,$$

and suppose that

$$k > \max\left(p + \frac{7}{2} + \frac{7}{2} C_4,\ k_6 + 3 + 2C_4,\ 4 + \frac{7}{2} C_4\right).$$

Suppose that $2 \leq m \leq (k - C_5)^2$ and P is a fixed point of order m on the segment $F_k F_{k+1}$. Write α for the abscissa of P. Then since

$$\alpha > a_k > (k - \tfrac{1}{2})\omega$$

we see that α satisfies (8.2), and, since m evidently satisfies (8.3), we have that there are a k' and an s such that

$$\alpha = X^*(0|k', m, s) < (k' + 1 + C_4)\omega.$$

We deduce that

$$k' > k - \frac{3}{2} - C_4 > \max\left(p + 2 + \frac{5}{2} C_4,\ k_6 + \frac{3}{2} + C_4,\ \frac{5}{2} + \frac{5}{2} C_4\right)$$

and

$$m \leq (k - c_5)^2 < (k' + \frac{3}{2} + c_4 - c_5)^2 < (k' - \frac{3}{2} - c_4)^2$$

and this implies that

$$X^{**}(t|k', m, s) = X^*(t|k', m, s)$$

with, in particular,

$$a_k < X^{**}(0|k', m, s) < a_{k+1} \quad .$$

Hence $k = k'$ and

$$\alpha = X^*(0|k, m, s).$$

It follows that there are exactly $\varphi(m)$ fixed points of order m on $F_k F_{k+1}$ and the set of these must be just the set of the P_{sm} introduced in Theorem 6. Since the latter satisfy (iii) of the enunciation of Theorem 10, there is no clash in our uses of the notation and Theorem 10 is proved.

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Tech. rec.

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